

Problem Session - Section 8.2

Main tools:
Definition of normal subgroup
Th 8.11 - criteria for a subgroup to be normal

16 p 253 K is a normal subgroup in G

$$|K| = 2$$

Prove that $K \subseteq Z(G)$

Pf $K = \{e, k\}$

Wanted: for every $a \in G$, $ak = ka$.

Let $a \in G$ $aka^{-1} \in K$ (Th 8.11)

$$aka^{-1} \in K$$

$$aka^{-1} = e$$

$$\text{or } aka^{-1} = k$$

$$ak = a$$

$$ak = ka$$

$$k = e$$

does not take place

20, p 253 a) N is normal in G

K is a subgroup in G

$NK = \{nk \mid n \in N, k \in K\}$ is a subgroup in G

Pf We use Th 7.11 as a criterion

① $a, b \in NK$ imply $ab \in NK$

② $a \in NK$ implies $a^{-1} \in NK$

$$\textcircled{1} \quad a = n_1 k_1 \quad b = n_2 k_2 \quad n_1, n_2 \in N, \quad k_1, k_2 \in K$$

$$n_1 k_1 n_2 k_2 = \underbrace{n_1}_{\in N} \underbrace{n_2}_{\in N} \underbrace{k_1 k_2}_{\in K} \in NK$$

$$k_1 n_2 = n_3 k_1$$

$$n_3 \in N$$

From the definition:

$$gN = N'g \quad g \in G$$

For any $g \in G, n \in N$
there is $n' \in N$ such that

$$gn = n'g$$

$$\textcircled{2} \quad a = nk \quad n \in N, \quad k \in K$$

$$a^{-1} = k^{-1} n^{-1} = n_4 k^{-1} \in NK$$

$$n_4 \in N \quad k^{-1} n^{-1} = n_4 k^{-1}$$

20, p 253 b) If both N and K are normal in G , then NK is normal in G

Pf Th 2.11 (3) allows us to check:

for every $a \in G, n \in N, k \in K$, we have $ank a^{-1} \in NK$

$$an = n_1 a \quad \text{with } n_1 \in N \quad ak = k_1 a \quad \text{with } k_1 \in K$$

$$\underline{ank a^{-1}} = \underline{n_1 a k a^{-1}} = n_1 k_1 a a^{-1} = n_1 k_1 \in NK$$

Pf By Th 2.11 (3), it suffices to prove that
for every $h \in H$, $k \in f(N)$ we have

$$\underline{hkh^{-1} \in f(N)}$$

For $h \in H$, by the surjectivity of f , we have $g \in G$ such that
 $h = f(g)$.

Also, $k \in f(N)$ means $k = f(n)$ with $n \in N$

$$\underline{hkh^{-1}} = f(g)f(n)(f(g))^{-1} = f(\underbrace{gng^{-1}}_N) \in \underline{f(N)}$$

f is a homomorphism

Since N is normal in G ,

by Th 2.11, $gng^{-1} \in N$.

26 p 254 H is the only subgroup of order n in G
Prove that H is a normal subgroup in G .

Pf Let $a \in G$. By 2.11, it suffices to show that
 $aHa^{-1} = H$

Easy to check: $aHa^{-1} = \{aha^{-1} \mid h \in H\} \subseteq G$
 is a subgroup in G (use Th 7.11)
 $aha^{-1} \neq ah'a^{-1}$ if $h \neq h'$ - easy, and
 implies $|H| = |aHa^{-1}|$

Therefore aHa^{-1} is also a subgroup in G of order n .

Since H is the only such subgroup, we have

$$\underline{H = aHa^{-1}}$$

27 p 254 N is normal in G iff $[ab \in N \text{ iff } ba \in N \text{ for every } a, b \in G]$
 iff/implies

Pf ① Let N be normal

$$\underline{ab \in N} \quad ab = n \in N \quad \underline{b = a^{-1}n = n_1 a^{-1}} \quad \underline{ba = n_1 \in N}$$

normality
 implies the existence
 of $n_1 \in N$

② Assume that $ab \in N$ iff $ba \in N$ for every $a, b \in G$.

By Th 8.11, it suffices to prove that $\underline{gng^{-1} \in N} \mid \underline{gng^{-1} \in N}$

for every $g \in G, n \in N$

Consider
 $g^n g^{-1}$

$$\text{Let } a = g^n \quad b = g^{-1}$$

$ba = g^{-1} g^n = n \in N$ implies $ab \in N$ by the assumption

$$\underline{ab = g^n g^{-1} \in N}$$

32 p254 Prove that $\text{Inn } G$ is a normal subgroup in $\text{Aut } G$

$\text{Aut } G = \{ \phi : G \rightarrow G \mid \phi \text{ is an isomorphism} \}$

the group operation is the composition of maps

An inner automorphism for $g \in G$

$$\begin{aligned} f_g : G &\longrightarrow G \\ a &\longmapsto g^{-1} a g \end{aligned}$$

$$\text{Inn } G = \{ f_g : G \rightarrow G \mid g \in G \}$$

Pf By Th 2.11, it suffices to check that for every $\sigma \in \text{Aut } G$,
and $g \in G$, we have
 $\sigma f_g \sigma^{-1} \in \text{Inn } G$

Let $a \in G$

$$\begin{aligned} (\sigma f_g \sigma^{-1})(a) &= \sigma f_g (\sigma^{-1}(a)) = \sigma (g^{-1} \sigma^{-1}(a) g) \\ &= \sigma(g^{-1}) \sigma(\sigma^{-1}(a)) \sigma(g) = \sigma(g)^{-1} a \sigma(g) = f_{\sigma(g)}(a) \end{aligned}$$

Proved: $\sigma f_g \sigma^{-1} = f_{\sigma(g)} \in \text{Inn } G$